Completing the squares

→ In P4, There are more than one of fundamental integration formulas involves the sum or difference of two squares. We can extend the application of these formulas by the use of the technique of completing the square.

→ Any quadratic polynomial (2nd degree polynomial) of general form,

$a x^2 + b x + C = \phi(x)$, Then

$$\phi(x) = a\left(x^2 + \frac{b}{a}x\right) + C$$

$$= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + C - \left(\frac{b}{2a}\right)^2$$

إشارة معامل x

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}$$

- Deviding by coeff. of x^2.
- Add & subtract the sq. of half the coeff. of x.
- Resolve into factors.

خطوات عملية الإكمال المربع

Express each polynomial as the sum or difference of squares:

(i) $4x^2 + 4x + 2 = 4\left(x^2 + x + \frac{1}{2}\right) = 4\left(x^2 + x + \frac{1}{4} + \frac{1}{2} - \frac{1}{4}\right)$

$= 4\left(x^2 + x + \frac{1}{4} + \frac{1}{4}\right) = 4\left(\left(x + \frac{1}{2}\right)^2 + \frac{1}{4}\right)$

$= \boxed{4\left(x + \frac{1}{2}\right)^2 + 1}$

(ii) $3x - x^2 = -\left(x^2 - 3x\right) = -\left(x^2 - 3x + \left(\frac{-3}{2}\right)^2 - \left(\frac{-3}{2}\right)^2\right)$

$= -\left(x^2 - 3x + \frac{9}{4}\right) + \frac{9}{4} = -\left(x - \frac{3}{2}\right)^2 + \frac{9}{4} = \boxed{\frac{9}{4} - \left(x - \frac{3}{2}\right)^2}$

(iii) $x^2 - 6x + 5 = x^2 - 6x + 9 + 5 - 9 = x^2 - 6x + 9 - 4$

$= \boxed{(x - 3)^2 - 4}$

(iv) $x^2 + 8x + 25 = x^2 + 8x + 4 + 25 - 16 = (x^2 + 8x + 16) + 9$

$= \boxed{(x + 4)^2 + 9}$

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Evaluate: (i) $I = \int \frac{dx}{4x^2+4x+2}$. (ii) $J = \int \frac{dx}{\sqrt{3x-x^2}}$.

(iii) $K = \int \frac{dx}{x^2-6x+5}$. (iv) $L = \int \frac{dx}{\sqrt{x^2+8x+25}}$.

Ans:

(i) $4x+4x+2 = 4\left(x^2+x+\frac{1}{2}\right) = 4\left(x^2+x+\frac{1}{4}+\frac{1}{2}-\frac{1}{4}\right)$

$= 4\left(\left(x+\frac{1}{2}\right)^2+\frac{1}{4}\right)$, Then

$$I = \int \frac{dx}{4\left(\left(x+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2\right)} = \frac{1}{4}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}$$

Put $x+\frac{1}{2} = u$, $\frac{1}{2} = a$

$$I = \frac{1}{4}\int \frac{du}{u^2+a^2} = \frac{1}{4}\cdot\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C$$

$$= \boxed{\frac{1}{2}\tan^{-1}(2x+1)+C}$$

(ii) $3x-x^2 = -(x^2-3x) = -\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right) = \frac{9}{4}-\left(x-\frac{3}{2}\right)^2$

$= \left(\frac{3}{2}\right)^2-\left(x-\frac{3}{2}\right)^2$, Then

$$J = \int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}} = \sin^{-1}\left(\frac{x-\frac{3}{2}}{3/2}\right)+C = \boxed{\sin^{-1}\left(\frac{2x-3}{3}\right)+C}$$

(iii) $x^2-6x+5 = x^2-6x+9+5-9 = (x-3)^2-(2)^2$, Then

$$K = \int \frac{dx}{(x-3)^2-(2)^2} = \boxed{\frac{1}{2}\coth^{-1}\left(\frac{x-3}{2}\right)+C}$$

(iv) $x^2+8x+25 = x^2+8x+16+25-16 = (x+4)^2+(3)^2$, Then

$$L = \int \frac{dx}{\sqrt{(x+4)^2+(3)^2}} = \boxed{\sinh^{-1}\left(\frac{x+4}{3}\right)+C}$$

Evaluate: (i) $I = \int \frac{x\,dx}{\sqrt{x^2+2x}}$, (ii) $J = \int \frac{x\,dx}{\sqrt{5+4x-x^2}}$

Ans:

(i) $x^2+2x = x^2+2x+1-1 = (x+1)^2-1$. Then

$$I = \int \frac{x+1-1}{\sqrt{(x+1)^2-1}}\,dx = \int \frac{x+1}{\sqrt{(x+1)^2-1}}\,dx - \int \frac{dx}{\sqrt{(x+1)^2-1}}$$

$$= \boxed{\sqrt{(x+1)^2-1} - \cosh^{-1}(x+1) + C} \#$$

(ii) $5+4x-x^2 = -(x^2-4x-5) = -(x^2-4x+4-5-4) = 9-(x-2)^2$, Then

$J = \int \frac{x}{\sqrt{9-(x-2)^2}}\,dx \Rightarrow$ Put $u = x-2 \Rightarrow du = dx$ & $x = u+2$, so

$$J = \int \frac{u+2}{\sqrt{9-u^2}}\,du = \frac{-1}{2}\int \frac{-2u}{\sqrt{9-u^2}}\,du + 2\int \frac{du}{\sqrt{9-u^2}}$$

$$= -\sqrt{9-u^2} + 2\sin^{-1}\left(\frac{u}{3}\right) + C$$

$$= \boxed{-\sqrt{9-(x-2)^2} + 2\sin^{-1}\left(\frac{x-2}{3}\right) + C} \#$$

Evaluate $I = \int \sqrt{\frac{a+x}{a-x}}\,dx$

Ans: you need to make a simplification to the integrand as follow,

$$\sqrt{\frac{a+x}{a-x}} * \frac{\sqrt{a+x}}{\sqrt{a+x}} = \frac{a+x}{\sqrt{a^2-x^2}} = \frac{a}{\sqrt{a^2-x^2}} + \frac{x}{\sqrt{a^2-x^2}}$$. Then

$$I = a\int \frac{dx}{\sqrt{a^2-x^2}} + -\int \frac{-x\,dx}{\sqrt{a^2-x^2}} = \boxed{a\sin^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2-x^2} + C} \#$$

Note: If the integrand consists of $\sqrt{\frac{ax+b}{cx+d}}$, simplification is made by multiplying the numerator and denominator by $\sqrt{ax+b}$.

إذا الجذر في البسط غير مقبول أثناء إجراء عملية التكامل

لأنه لما يكون الجذر عندي في المقام نقدر نخضع صورة يكون تكاملها هو أحد الدوال العكسية مع تقسيم البسط

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6. Integration by Partial fraction.

We know that $\boxed{\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}}$ ⇒ Combining fractions over a common denominator, so

We can say that a partial fractions decomposition for $\boxed{\frac{5}{6}}$ is $\boxed{\frac{5}{6}=\frac{1}{2}+\frac{1}{3}}$

How to find a partial fractions decomposition?

لما عندما تكون ال integrand ⇐ rational function وتكتب $\boxed{\frac{P(x)}{q(x)}}$

للحصول على ال Partial fraction decomposition لل $\boxed{\frac{P(x)}{q(x)}}$

يجب أن + تكون درجة $P(x)$ أقل من درجة $q(x)$.
+ تكون $q(x)$ قابلة للتحليل.

وهنشوف خطوات إيجاد ال Partial fraction decomposition في مثال

$$\frac{6}{x^2-1}$$

(1) • $\boxed{\text{Begin with factoring denominator}}$ (تحليل المقام)

$$\frac{6}{x^2-1}=\frac{6}{(x-1)(x+1)}$$

(2) • $\boxed{\text{Assume constants } A, B}$ so that

$$\frac{6}{(x-1)(x+1)}=\frac{A}{(x-1)}+\frac{B}{(x+1)}=\frac{A(x+1)+B(x-1)}{(x-1)(x+1)}, \text{ Then}$$

(3) $\boxed{\text{common denominator}}$

$A(x+1)+B(x-1)=6$ ⇒ (4) $\boxed{\text{finding constants}}$

Put $x=-1 \Rightarrow -2B=6 \Rightarrow B=-3$
$x=1 \Rightarrow 2A=6 \Rightarrow A=3$

بنختار قيم (x) بحيث إنها تدي صفر مع أحد الثوابت A أو B في كل مرة

Then $\boxed{\frac{6}{x^2-1}=\frac{3}{(x-1)}-\frac{3}{(x+1)}}$ #

Find Partial fraction decomposition for

(1) $$\frac{2x-3}{x^3+x} = \frac{2x-3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}$$

ملاحظة

هنا عند تحليل المقام .. ويظهر قوس من الدرجة الثانية وغير قابل للتحليل تكون البسط في Polynomial من الدرجة الأولى ← $Bx+C$

Then, $2x-3 = A(x^2+1)+(Bx+C)x$

Put $x=0 \Rightarrow \boxed{-3 = A}$,

Put $x = i$, Then

أو بأي رقم وتقارن المعاملات

$$2i - 3 = -3\underbrace{(i^2+1)}_{0} + (Bi+C)i$$

$$2i - 3 = Ci - B \Rightarrow \boxed{C = 2 \ , \ B = 3}$$, Then

$$\boxed{\frac{2x-3}{x^3+x} = \frac{-3}{x} + \frac{3x+2}{x^2+1}}$$

(2) $$\frac{x^2+3}{x^3+x^2-x-1} = \frac{x^2+3}{x^2(x+1)-(x+1)} = \frac{x^2+3}{(x+1)(x^2-1)} = \frac{x^2+3}{(x+1)^2(x-1)}$$

لاحظ الفرض هنا عامل إزاي؟

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} = \frac{\quad\quad}{(x-1)(x+1)^2}$$

$$A(x+1)^2 + B(x-1)(x+1) + C(x-1) = x^2+3$$

Put $x=-1 \Rightarrow -2C = 4 \Rightarrow \boxed{C = -2}$

$x = 1 \Rightarrow 4A = 4 \Rightarrow \boxed{A = 1}$

$x = 0 \Rightarrow 1 - B + 2 = 3 \Rightarrow \boxed{B = 0}$

$$\boxed{\frac{x^2+3}{x^3+x^2-x-1} = \frac{1}{x-1} + \frac{-2}{(x+1)^2}}$$

Note for $\boxed{\dfrac{f(x)}{(x-1)(x-2)^3(x^2+x+1)^2}}$ & $f(x)$ is Polynomial of degree less than 8 (degree of denominator), Then the Partial fraction decomposition for it will be:

$$\boxed{\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3} + \frac{Ex+F}{x^2+x+1} + \frac{Gx+H}{(x^2+x+1)^2}}$$

ثم نقوم بإيجاد الثوابت
A, B, C, D, E, F, G, H

Find:

درجة البسط (0) < درجة المقام (2)

أكبر (Power of x)

(1) $I = \int \frac{dx}{1-x^2} = \boxed{\coth^{-1} x + C}$

Ans: By partial fraction, $\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$

$= \frac{A(1+x) + B(1-x)}{(1-x)(1+x)}$, Then

$A(1+x) + B(1-x) = 1 \Rightarrow$ Put: $x = 1 \Rightarrow A = \frac{1}{2}$

$x = -1 \Rightarrow B = \frac{1}{2}$

$\therefore I = \frac{1}{2}\int \frac{dx}{1+x} + \frac{1}{2}\int \frac{dx}{1-x} = \frac{1}{2}\ln|1+x| + \frac{-1}{2}\ln|1-x| = \boxed{\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C}$

$\coth^{-1} x$

(2) $\int \frac{x+7}{x^2-x-6}\,dx$ → درجة البسط (1) < درجة المقام (2)

Ans: By partial fraction:

$\frac{x+7}{x^2-x-6} = \frac{x+7}{(x+2)(x-3)} = \frac{A}{(x+2)} + \frac{B}{(x-3)} = \frac{A(x-3) + B(x+2)}{(x+2)(x-3)}$, Then

$A(x-3) + B(x+2) = x+7$,

Put: $x = 3 \Rightarrow 5B = 10 \Rightarrow \boxed{B = 2}$, Then

$x = -2 \Rightarrow -5A = 5 \Rightarrow \boxed{A = -1}$

$I = \int \frac{2}{(x-3)}\,dx - \int \frac{dx}{x+2} = 2\ln|x-3| - \ln|x+2| + C$

$= \boxed{\ln\frac{(x-3)^2}{|x+2|} + C}$

(3) $I = \int \frac{2x^3 - 4x - 8}{(x^2-x)(x^2+4)}\,dx$ → درجة البسط (3) < درجة المقام (4)

Ans: By partial fraction,

$\frac{2x^3-4x-8}{(x^2-x)(x^2+4)} = \frac{2x^3-4x-8}{x(x-1)(x^2+4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4}$, Then

$A(x-1)(x^2+4) + Bx(x^2+4) + (Cx+D)x(x-1) = 2(x^3 - 2x - 4)$

Put: $x = 1 \Rightarrow 5B = -10 \Rightarrow \boxed{B = -2}$,

$x = 0 \Rightarrow -4A = 2 \times -4 \Rightarrow \boxed{A = 2}$,

$x = 2i \Rightarrow (2Ci + D)\,2i(2i-1) = 2(-8i - 4i - 4)$

$(2Ci + D)(-4 - 2i) = -24i - 8$

$-8Ci + 4C - 4D - 2Di = -24i - 8 \Rightarrow$, $8C + 2D = 24 \Rightarrow 4C + D = 12 \rightarrow (1)$

$4C - 4D = -8 \Rightarrow C - D = -2 \rightarrow (2)$

$5C = 10 \Rightarrow \boxed{C = 2}$

$\boxed{D = 4}$

Then,

$$I = \int \left[\frac{2}{x} - \frac{2}{x-1} + \frac{2x+4}{x^2+4} \right] dx$$

$$= 2\ln|x| - 2\ln|x-1| + \int \frac{2x}{x^2+4} dx + 4\int \frac{dx}{x^2+4}$$

$$= 2\ln|x| - 2\ln|x-1| + \ln|x^2+4| + 2\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= \boxed{\ln\left[\frac{x^2(x^2+4)}{(x-1)^2}\right] + 2\tan^{-1}\left(\frac{x}{2}\right) + C}$$

(4) $I = \int \frac{2x^5 - 5x}{(x^2+2)^2} dx$

Ans: Here, the degree of numerator (5) is greater than degree of denominator (4) So, we make long division. So

$$\begin{array}{r|l} & 2x \\ \hline x^4+4x^2+4 & 2x^5 \qquad\qquad -5x \\ & \underline{2x^5 \mp 8x^3 \mp 8x} \\ & (-8x^3 - 8x - 5x) \text{ rest} \end{array}$$

$$\frac{2x^5 - 5x}{(x^2+2)^2} = 2x - \left(\frac{8x^3+13x}{(x^2+2)^2}\right) \rightarrow ①$$

$$\frac{8x^3+13x}{(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2} = \frac{(Ax+B)(x^2+2) + Cx+D}{(x^2+2)^2}, \text{ Then}$$

$$(Ax+B)(x^2+2) + Cx + D = x(8x^2+13)$$

put $x = \sqrt{2}\,i \Rightarrow \sqrt{2}C\,i + D = \sqrt{2}\,i * -3 \Rightarrow \boxed{D=0}, \boxed{C=-3}$

$x = 0 \Rightarrow 2B = 0 \Rightarrow \boxed{B=0}$

$x = 1 \Rightarrow 3A - 3 = 21 \Rightarrow 3A = 24 \Rightarrow \boxed{A=8}$

Then: $I = \int 2x\, dx - \int \frac{8x}{x^2+2} dx + \int \frac{3x}{(x^2+2)^2} dx$

$$I = \boxed{x^2 - 4\ln(x^2+2) - \frac{3}{2(x^2+2)} + C} \;\#$$

خد بالك

يجب أن تكون ع دراية بأنه قد توجد أكثر من طريقة لإجراء التكامل وهنا تظهر ميزتك في اختيار الطريقة الأبسط والتي هتقدر تحددها بحل تمارين كثيرة

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G.M.E 2015

End of chapter.